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The stability of plane Couette flow with viscosity stratification

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The stability of plane Couette flow is examined for liquids in which the viscosity varies with depth. Under suitable conditions, the flow may be stable or unstable at small Reynolds numbers for disturbances with wavelengths long compared with the liquid depth.

A mechanism which may be stabilizing or destabilizing is found to derive from the role of diffusion in the neighbourhood of a 'critical layer', where the liquid velocity equals the phase velocity of a wave-like disturbance. This mechanism, which requires a viscosity *gradient*, is quite distinct from that found by Yih (1967) for the case of a viscosity *discontinuity*. An example is considered, for which these two mechanisms may be of comparable importance.

1. Introduction

Yih (1967) has examined the stability of plane Couette–Poiseuille flow of two superposed liquid layers of different viscosities but equal densities. He found that such flows can be unstable at arbitrarily small Reynolds numbers, the unstable disturbances having wavelengths large compared with the liquid depths. As for liquid films flowing down an inclined plane under gravity—see Benjamin (1957) and Yih (1954, 1963)—the instability mechanism derives from inertia forces, which, though small compared with the viscous forces, nevertheless govern the stability of the system. A similar mechanism operates in the recent work of Kao (1968) on the stability of two-layer flow down an inclined plane. The present paper examines plane Couette flow in which the viscosity may vary continuously as well as discontinuously with depth; and a further mechanism is revealed which may promote stability or instability. This mechanism does not involve inertia forces, but depends for its existence on the presence of a viscosity *gradient* in the fluid. It is therefore quite distinct from that found by Yih.

In a recent paper, Craik & Smith (1968) examined the stability of free-surface flows with continuous viscosity stratification. Their analysis was based on the assumption that the viscosity μ of each liquid particle remains constant throughout its motion: that is,

$$D\mu/Dt' = 0,$$

where D/Dt' denotes the total time derivative. This assumption is likely to be a good approximation when the effects of diffusion are slight. However, if the phase velocity of a small periodic disturbance equals the velocity of the liquid at some depth, the linearized form of the above equation is singular at this depth. The role of such singularities was not considered by Craik & Smith, since their analysis concerned only 'surface waves' with phase velocities greater than the maximum velocity of the liquid. In contrast, the present work examines a physical mechanism which is closely related to these singularities.

In the vicinity of the 'critical layer' where the liquid velocity equals the phase velocity of a wave-like disturbance, it is apparent that diffusion of viscosity (by whatever means) may have a significant effect; and that the inclusion of appropriate diffusive terms will remove the singularity mentioned above. Such a procedure is like that used to examine the stability of flows at large Reynolds numbers (see, for example, Lin 1955), where the 'critical-layer' singularity of the inviscid equation of motion may be removed by the inclusion of viscous terms. Even more directly relevant is the work of Lees & Lin (1946) on the stability of compressible boundary layers; there, the inviscid equation of motion is singular at the critical layer and so also is the heat equation when thermal diffusion is neglected in comparison with convection. The singularity of the latter equation is resolved by considering the role of thermal diffusion in the vicinity of the critical layer.

2. The stability problem

Liquid is confined between two rigid parallel plane boundaries, a distance h apart. One boundary is fixed and the other is constrained to move in its own plane with a constant velocity V. The liquid density ρ is assumed to be constant; accordingly, gravity has no dynamical effect and the orientation in space of the plane boundaries is arbitrary. However, for brevity, the distance normal to the fixed boundary will be called the 'depth'. In the absence of any perturbation, the viscosity of the liquid is a function of this depth only.

All quantities are made dimensionless with respect to the depth h, the velocity V and the density ρ. Dimensionless co-ordinates (x, y) are chosen such that the x-axis is parallel to the direction of motion and the y-axis is normal to the plane boundaries. The fixed boundary is located at $y = 0$ and the moving boundary at $y = 1$. Dimensionless time is denoted by t and the dimensionless velocity components and pressure by (u, v) and p respectively.

A dimensionless viscosity m is defined as

$$m(x, y, t) = \mu(x, y, t)/\mu_0,$$

where $\mu(x, y, t)$ is the actual viscosity and μ_0 is some constant viscosity which is characteristic of the liquid as a whole. Attention is restricted to plane Couette flow, for which there is no pressure gradient in the flow direction. Therefore, in the absence of any perturbation, there is a uniform shear flow for which the primary velocity profile $\overline{u}(y)$ and viscosity distribution $\overline{m}(y)$ satisfy an equation of the form

$$\overline{m}\,\overline{u}' = \text{constant}\,(=1), \tag{2.1}$$

where the prime denotes d/dy. Without loss of generality, the characteristic viscosity μ_0 may be chosen so that this constant equals unity. A Reynolds number is defined as $R \equiv \rho Vh/\mu_0$.

In the absence of any diffusive agent, each liquid particle has a constant viscosity, i.e. $Dm/Dt = 0$. However, as explained in the introduction, the influence of a diffusive mechanism, *however weak*, may be significant for the waves to be discussed. For simplicity, we assume that

$$Dm/Dt = K\nabla^2 m, \tag{2.2}$$

where K is a dimensionless coefficient of diffusivity. [In practice, this equation may not be strictly correct. For example, if changes in viscosity are the direct result of changes in temperature T, the appropriate relations would be the heat equation for T, together with an equation denoting the dependence of viscosity on temperature. However, such modifications only introduce additional terms in the equation for m, which turn out to be unimportant in the present context; the effect of diffusion is restricted to the vicinity of the 'critical layer' (see § 4), where the term $K(\partial^2 m/\partial y^2)$ is dominant.]

We now suppose that the primary flow experiences a small two-dimensional disturbance which is periodic in the x-direction and which is governed by a perturbation stream function

$$\psi(x,y,t) = f(y)\,\eta(x,t), \quad \eta(x,t) \equiv e^{i\alpha(x-ct)},$$

such that the velocity components are

$$u = \overline{u}(y) + f'(y)\,\eta, \quad v = -i\alpha f(y)\,\eta.$$

Here α is a real dimensionless wave-number and c is a dimensionless wave velocity which may be complex. The corresponding dimensionless viscosity is

$$m = \overline{m}(y) + \hat{m}(y)\,\eta.$$

We now make the assumptions

$$\alpha^2 \ll 1; \quad \alpha R \ll 1, \quad \alpha R\,|c| \ll 1, \tag{2.3a–c}$$

of which (2.3a) requires the wavelength of the disturbance to be large compared with the depth h, and (2.3b, c) require viscous forces to be large compared with inertia forces in the equations of motion. With these assumptions, Craik & Smith (1968) have obtained

$$(\overline{m}f'' + \overline{u}'\hat{m})'' = 0 \tag{2.4}$$

as a first approximation to the linearized equations of motion. To the same approximation (i.e. on neglecting terms in α^2 by virtue of (2.3a)), (2.2) yields the linearized result

$$\overline{m}'f = (\overline{u} - c)\,\hat{m} + (iK/\alpha)\,\hat{m}''. \tag{2.5}$$

The boundary conditions to be satisfied by $f(y)$ are

$$f(0) = f'(0) = f(1) = f'(1) = 0. \tag{2.6}$$

3. The eigenvalue equation

We first assume that the diffusive term of (2.5) may be neglected and K set equal to zero. Then (2.4) and (2.5) yield the approximate results (see Craik & Smith 1968, §5)

$$\left.\begin{aligned}\hat{m} &= \frac{-\bar{u}''f}{\bar{u}'^2(\bar{u}-c)},\\ (\bar{u}-c)f''-\bar{u}''f &= \bar{u}'(\bar{u}-c)(Ay+B),\end{aligned}\right\} \qquad (3.1a,b)$$

where A and B are constants of integration to be determined by the boundary conditions. Further, the general solution of (3.1b) is easily shown to be

$$f(y) = Cf_1(y)+Df_2(y)+I(y),$$

where
$$f_1 = \bar{u}-c, \quad f_2 = (\bar{u}-c)\int_0^y [\bar{u}(y_1)-c]^{-2}\,dy_1, \qquad (3.2)$$

$$I = \tfrac{1}{2}(\bar{u}-c)\left\{Ay+\tfrac{1}{2}By^2-B\int_0^y [\bar{u}(y_1)-c]^{-2}\int_0^{y_1}[\bar{u}(y_2)-c]^2\,dy_2\,dy_1\right\}.$$

Here I is a particular integral of (3.1b) and f_1, f_2 are solutions of the associated homogeneous equation.

The boundary conditions (2.6) determine the constants A, B, C, D and yield the eigenvalue equation for c, namely

$$\left[(1-c)^2-\int_0^1(\bar{u}-c)^2\,dy\right]\left[1-c^2\int_0^1(\bar{u}-c)^{-2}\,dy\right]$$
$$= (1-2c)\left[\tfrac{1}{2}-\int_0^1(\bar{u}-c)^{-2}\int_0^y[\bar{u}(y_1)-c]^2\,dy_1\,dy\right]. \qquad (3.3)$$

If c is real and $0 < c < 1$, there is a critical layer at some depth y_c in the interval $0 < y < 1$ where $c = \bar{u}(y_c)$. Those integrands of (3.3) which involve negative powers of $(\bar{u}-c)$ are then singular at y_c. When c is real, (3.3) may be rewritten as

$$\left\{(1-c)^2\left[\int_0^{y_c}(\bar{u}-c)^2\,dy-c^2\right]+c^2\int_{y_c}^1(\bar{u}-c)^2\,dy\right\}\int_0^1(\bar{u}-c)^{-2}\,dy$$
$$= \int_0^1(\bar{u}-c)^2\,dy-(1-c)^2+(1-2c)\left\{\tfrac{1}{2}+\int_0^1(\bar{u}-c)^{-2}\int_y^{y_c}[\bar{u}(y_1)-c]^2\,dy_1\,dy\right\}, \qquad (3.4)$$

in which only the integral
$$\int_0^1(\bar{u}-c)^{-2}\,dy \qquad (3.5)$$
is singular.

Now it is shown in the following section that this integral may be evaluated by indenting *under* the singularity at y_c in the complex y-plane. Thus

$$\int_0^1(\bar{u}-c)^{-2}\,dy = \mathscr{P}\int_0^1(\bar{u}-c)^{-2}\,dy-\frac{i\pi\bar{u}_c''}{\bar{u}_c'^3}, \qquad (3.6)$$

where $\mathscr{P}$ denotes the principal part of the integral and the subscript c denotes evaluation at y_c. But, when c is real, the only imaginary contribution to (3.4) derives from the latter term of (3.6). Consequently, if real values of c are to exist, *either* $\bar{u}_c''$ is zero *or*

$$(1-c)^2\left[\int_0^y (\bar{u}-c)^2\,dy - c^2\right] + c^2\int_{y_c}^1 (\bar{u}-c)^2\,dy = 0. \tag{3.7}$$

However, since $\bar{u}(y)$ increases monotonically in the interval $(0,1)$, we have the inequalities

$$(\bar{u}-c)^2 < c^2 \qquad (0 < y < y_c),$$
$$(\bar{u}-c)^2 < (1-c^2) \quad (y_c < y < 1),$$

and it follows that the left-hand side of (3.7) is always *negative*. Accordingly, (3.7) can never be satisfied and we have shown that *a necessary condition for the existence of a neutral disturbance with* $0 < c < 1$ *is that* $\bar{u}_c'' = 0$. A corollary of this result is that, *if* $\bar{u}''(y)$ *is non-zero for all* y *in the interval* $(0,1)$, *there can be no neutral disturbance with* $0 < c < 1$. (It should be remembered, however, that these results hold only within the range of validity of the present approximations.) Clearly, when $\bar{u}_c''$ is non-zero, disturbances must be amplified or damped. In order to determine which is the case, the eigenvalue equation for c must be solved for particular velocity profiles $\bar{u}(y)$. Such an example is discussed in §5. First, however, we must examine the role of diffusion in the vicinity of the critical layer.

4. The 'diffusive' solutions

The role of diffusion may not be neglected in the vicinity of the critical layer y_c where $\bar{u}(y_c)$ equals the phase velocity c. By eliminating $\hat{m}$ between results (2.4) and (2.5), we obtain

$$\left(\bar{u}-c+\frac{iK}{\alpha}\frac{d^2}{dy^2}\right)\left(\frac{f''}{\bar{u}'^2}-\frac{Ay+B}{\bar{u}'}\right) = \frac{\bar{u}''f}{\bar{u}'^2}, \tag{4.1}$$

where A and B are the constants of integration introduced previously. In order to obtain solutions for f which are valid near y_c, we follow the approach of Lin (1955, p. 34) and seek two solutions of the form

$$f_{3,4} = \exp\left[\pm\tfrac{2}{3}(i\alpha\bar{u}_c'/K)^{\frac{1}{2}}(y-y_c)^{\frac{3}{2}}\right]\{g_0(y) + (K/\alpha)^{\frac{1}{2}}g_1(y) + \ldots\}.$$

It is readily found that the leading term is

$$g_0(y) = \text{constant} \times (y-y_c)^{-\frac{5}{4}},$$

in agreement with the case examined by Lin. Clearly, when $(K/\alpha)^{\frac{1}{2}}$ is small, the first approximation to the 'diffusive solutions' $f_{3,4}$ is precisely analogous to that for the 'viscous solutions' obtained by Lin. A consequence of this analogy is that Lin's arguments concerning the evaluation of integrals such as (3.5) are directly applicable. Therefore the appropriate path of integration in the complex y-plane is that obtained by indenting *under* the singularity at the critical point y_c, as was done in deriving result (3.6). The results of Lin (1955, §8) distinguish between regions of the complex y-plane in which inviscid solutions yield a valid first approximation and regions where the influence of viscosity cannot be neglected. In the present context, these results show that the role of *diffusion* cannot be neglected in that sector of the complex y-plane denoted by

$$\tfrac{1}{6}\pi \leqslant \arg(y-y_c) \leqslant \tfrac{5}{6}\pi,$$

even in the limit $K \to 0$; and that diffusion is also important in a region surrounding the critical point y_c with radius of order $(K/\alpha)^{\frac{1}{3}}$.

5. The stability of superposed liquid layers

We now consider the particular example of plane Couette flow of two superposed liquid layers. This case was examined by Yih (1967), who discovered an instability mechanism which is due to the difference in viscosities of the two layers. Yih considered the viscosities of the two liquids to be constant; in which case the velocity profiles for plane Couette flow are linear in either liquid, but the two layers have different velocity gradients. Accordingly we define $\overline{u}(y)$ to be

$$\left.\begin{aligned} \overline{u}(y) &= Uy/d & (0 \leqslant y \leqslant d) \\ &= U+(1-U)(y-d)/(1-d) & (d \leqslant y \leqslant 1) \end{aligned}\right\} \quad (5.1)$$

where the interface between the layers is at $y = d$ $(0 < d < 1)$ and the dimensionless velocity of the interface is U (see figure 1).

The curvature of this velocity profile is zero everywhere except at the interface $y = d$; and, apart from the exceptional case where the critical layer is at $y = d$, the condition $\overline{u}_c'' = 0$ is satisfied whenever $0 < \operatorname{Re}\{c\} < 1$. It follows that the necessary condition of § 3 for the existence of a neutral disturbance with $0 < c < 1$ is satisfied (subject to the condition that inertia terms are negligible). On substituting for $\overline{u}(y)$ in the eigenvalue equation (3.4) and indenting under the singularity at y_c, it is found that

$$c-U = \frac{2U(U-1)(U-d)}{-3U^2+2U(1+d)+(1-d)^2}, \quad (5.2)$$

which may be shown to agree with result (34) of Yih's paper. Since c is real, the disturbance is neutrally stable to this approximation.

If the viscosities of the two liquids differ, the dimensionless velocity gradient in one layer must exceed unity and that in the other layer must be less than unity. Therefore, without loss of generality, we may always choose U and d to be such that $0 < U/d < 1$, since this simply requires the liquid adjacent to $y = 0$ to be more viscous than that adjacent to $y = 1$. It follows from result (5.2) that $U < c < 1$; consequently the critical layer y_c must lie in the less viscous liquid.

By including inertia terms which were neglected in the above approximation, Yih found a higher approximation for c which has a non-zero imaginary part c_i whose sign determines the stability of the disturbance. However, at this point we depart from Yih's treatment and assume that inertia forces remain negligible; also, we allow the viscosity of the less viscous liquid to *vary slightly with depth* instead of being strictly constant. Such continuous variation of viscosity produces a small curvature of the velocity profile at the critical layer; and, since $\overline{u}_c''$ is no longer zero, the disturbance cannot be neutrally stable. For simplicity, we assume that the consequent deviation of the velocity profile from that of (5.1) is sufficiently small for result (5.2) to provide a good approximation for the real part c_r of the phase velocity c. We also assume that the imaginary part c_i of c is sufficiently small to permit the neglect of terms which are $O(|c_i|^2)$ in the eigenvalue equation (3.3).

The eigenvalue equation then has real and imaginary parts, the real part of which gives result (5.2) with c replaced by c_r. The imaginary part yields an expression of the form

$$c_i = (\pi \bar{u}_c''/\bar{u}_c'^3)\, G(U, d, c_r), \tag{5.3}$$

where the right-hand side derives from the imaginary contribution to the integral (3.6). The function $G(U, d, c_r)$ is a complicated algebraic expression which is omitted for brevity. Since $\bar{u}_c'$ equals $(1-U)/(1-d)$ to good approximation and c_r is given by result (5.2), values of $(c_i/\bar{u}_c'')$ may be calculated for particular values of U and d. Since the function G depends only on the velocity profile (5.1), and not on the small deviations from it owing to changes in viscosity, the sign of $\bar{u}_c''$ is of crucial importance.

Values of $(c_i/\bar{u}_c'')$ were computed by Mr C. D. McArthur for several values of U and d, and the results are shown in figures 2(*a*) and (*b*). From these it is seen that $(c_i/\bar{u}_c'')$ is always negative, and that $|c_i/\bar{u}_c''|$ is largest when the two liquids are of about equal depths and when U/d is about 0·5–0·6. Clearly, *the disturbance is amplified if $\bar{u}_c''$ is negative and damped if $\bar{u}_c''$ is positive.* This conclusion is illustrated in figure 1.

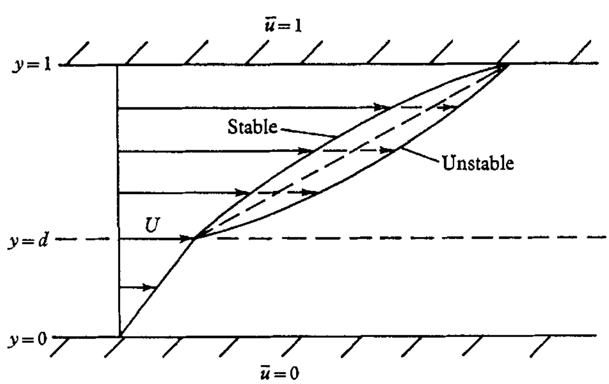

FIGURE 1. Sketch of flow configuration.

The above result sheds light on the case of two miscible liquids which, owing to molecular diffusion, do not have a sharp interface, but whose velocity profile is still close to (5.1) over most of the depth. Because of mixing, the viscosity of the less viscous liquid will *increase* as the 'interface' is approached; and this corresponds to the case $\bar{u}_c'' > 0$, for which diffusion tends to *stabilize* the disturbance. However, it was pointed out by Yih (1967, p. 351) that, since the critical layer always lies at some distance from the interface, it will fall outside the region in which significant viscosity gradients occur. If this is so, Yih's mechanism for instability will be dominant, and the above damping action due to a viscosity gradient will be unimportant. Nevertheless, it appears that *very small* viscosity gradients must be regarded as significant.

To examine the relative importance of the two mechanisms, we combine result (5.3) with Yih's equation (41) to obtain

$$c_i = \alpha R\, J(m_1, n) + (\pi \bar{u}_c''/\bar{u}_c'^3)\, G(U, d, c_r), \tag{5.4}$$

where c_r is given by (5.2). Here $J(m_1, n)$ is a function of the viscosity ratio m_1 and the depth ratio n of the two liquids; and this function is shown in figures 2 (*a*), (*b*) of Yih's paper for plane Couette flow. As an illustration we consider the special case $d = 0{\cdot}5$, $U/d = 0{\cdot}1$, which, in Yih's notation, corresponds to $n = 1$, $m_1 = 19$.

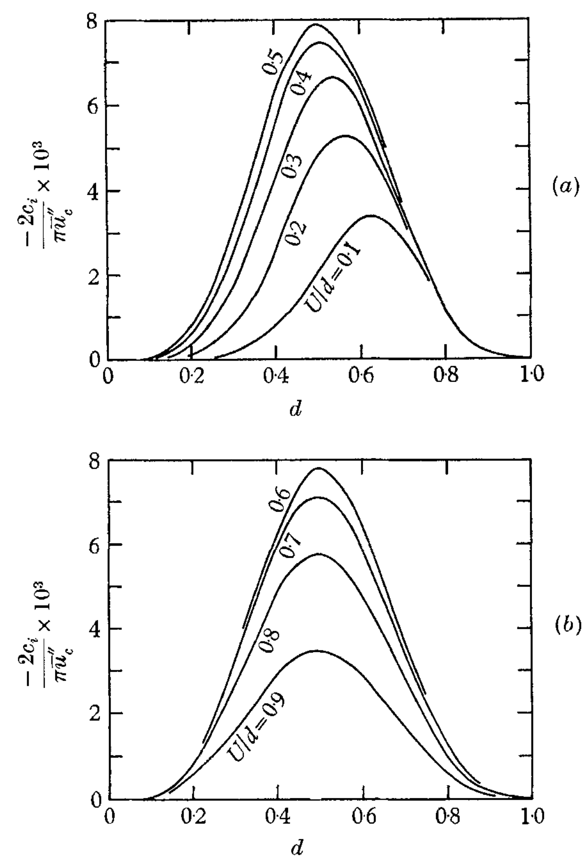

FIGURE 2 (*a*), (*b*). Curves of $-(2c_i/\pi \bar{u}_c'') \times 10^3$ *vs.* d for various constant values of U/d.

From Yih's figure 2 (*a*), the appropriate value of $J(m_1, n)$ is found to be about $7{\cdot}5 \times 10^{-4}$. This is close to the largest recorded value of $J(m_1, n)$ and it therefore denotes one of his most unstable configurations. The magnitude of the second term of (5.4) is seen from figure 2 (*a*) of the present paper to be $-(1{\cdot}05\pi \bar{u}_c'' \times 10^{-3})$. A comparison of the two terms shows that, when $\bar{u}_c''$ equals about $0{\cdot}23\alpha R$, the destabilizing effect of Yih's mechanism is balanced by a stabilizing effect due to diffusion and the disturbance is neutrally stable. Clearly, since αR is assumed to be small, the two mechanisms may be of comparable importance, even for very small values of $\bar{u}_c''$.

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